Midterm Semester-13/14

Solution 1. The auxiliary equation of the given ODE:

$$m^2 - 4m + 13 = 0$$

and the roots of this equation are $2 + 3i$ and $2 - 3i$.

Therefore the general solution of ODE is

$$x = e^{2t}(c_1 cos3t + c_2 sin3t),$$

where $t > 0$ and c_1, c_2 are arbitrary constants.

Solution 3. The given equation can be written as

$$-\frac{1}{y^3}\frac{dy}{dt} + 9\frac{1}{y^2} = 1.$$

Put $\frac{1}{y^2} = z$. Then the above ODE becomes

$$\frac{dz}{dt} + 18z = 2$$

and integrating factor of this equation is e^{18t}.

Integrating we have

$$\frac{1}{y^2} = ce^{-18t} + \frac{1}{9}.$$

(a) For $y(0) = \alpha > 0$, the solution is

$$\frac{1}{y^2} = (\frac{1}{\alpha^2} - \frac{1}{9})e^{-18t} + \frac{1}{9}.$$

(b) From the existence and uniqueness theorem (Picard Lindelof theorem), we have $y \equiv 0$.
(c) For (a) part,

$$y(t) = \frac{1}{\sqrt{(\frac{1}{\alpha^2} - \frac{1}{9})e^{-18t} + \frac{1}{9}}}.$$

Therefore $\lim_{t\to\infty} y(t) = 3$ and for (b) $\lim_{t\to\infty} y(t) = 0$.

Solution 4. (a) Here

$$y_1(t) = 15(t + e)^2$$

,

$$y_2(t) = -4(t + e)^3 ln(t + e)$$

are two solutions of the given ODE.

Therefore Wronksian $W(y_1, y_2)$ of y_1 and y_2 is

$$W(y_1, y_2) = y_1 y_2^{'} - y_2 y_1^{'} = 60(t + e)^4(-1 - ln(t + e)).$$

Also for $t > 0$, $W(y_1, y_2) \neq 0$. Hence

$$p(t) = \frac{W^{'}(t)}{W(t)} = \frac{-60(5 + 4ln(t + e))}{(t + 3)(1 + ln(t + e))}.$$

(b) To solve this, we will use the following theorem:
Suppose that q and $\tilde{q}$ are positive functions with $q > \tilde{q}$. Let y be a nontrivial solution of the differential equation

$$y^{''} + qy = 0$$

and let $\tilde{y}$ be a nontrivial solution of the differential equation

$$\tilde{y}^{''} + \tilde{q}\tilde{y} = 0.$$

Then y vanishes at least once between any two successive zeros of $\tilde{y}$.

Consider the ODE:

$$y^{''} - 2y = 0.$$

Then the general solution of the above ODE is

$$y = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

Now y vanishes at most one point. The given ODE is:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = 0. \tag{1}$$

Suppose the nontrivial solution x of eq. (1) vanishes in three different points on $[0, \infty)$, then from the above theorem, y must vanishes at least 2 different points, which is not possible. Hence the cardinality of $Z = \{a \in [0, \infty) : x(a) = 0\}$ is at most 2.